Verifying Probabilistic Correctness in Isabelle with pGCL

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Outline

- Stochastic Behaviour in Systems
  - Functional vs. Probabilistic Verification
  - pGCL in Isabelle/HOL
  - Example: Lattice-Lottery Scheduler
Sources of Uncertainty

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- Device failure.

Some things are simply too complex to model:
- A modern processor.
Classical nondeterminism is the ultimate in pessimism: Anything that *can* happen *will* happen.

If we know how events are distributed, we can do better. Probabilistic models are a halfway-house between full nondeterminism and full predictability.

Probabilistic guarantees are relevant both for security, and for reliability.

Our current work is on probabilistic security guarantees.
Why is this relevant in systems?

Feed a secret string and a guess to `strcmp`:

```
0
0.01
0.02
0.03
0.04
24 24.5 25
```

This is a side-channel, which exposes the secret. How bad is it? How can we mitigate it? How will it behave in a larger system?
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![Graph showing probability density vs. response time](image)

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\[
\wp ((r, \tau) := \text{strcmp}(g, s); \\
g := \text{cleverness}(r, \tau, g)) (g = s) \leq 2^{-100}
\]

Formulating this rigorously is the subject of our existing work. Mechanising this work in Isabelle/HOL ensures our reasoning is sound, and scalable to large problems. We use pGCL, an extension of Dijkstra's GCL with probability.
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Judgements on Programs

How do we interpret this?

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This relates a program to an annotation. If \( x = 0 \) holds before, then \( y = x \) holds afterwards. Is \( x = 0 \) maximal?

\{ x = 0 \lor x = 1 \} is maximal, it is the weakest precondition of \( \{ y = x \} \).
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\[ \wp \ a \ Q \equiv \sup \{ P | P \ a \ Q \} \]

\[ \{ R \} \leq \{ S \} \equiv R \vdash S \equiv \forall s. \ R \ s \rightarrow S \ s \]
Nondeterminism

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Algebraically: $\wp (a \sqcap b) Q = \wp a Q \cap \wp b Q$

Thus $P = \{x = 0 \lor x = 1\} \cap \{x = 0\} = \{x = 0\}$.

We are treating annotations as sets.
Quantitative Predicates

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Identify a set with its selector: \( \langle P \rangle \, s \equiv 1 \) if \( s \in P \) else 0.

We can still order these: \( \langle P \rangle \leq \langle Q \rangle \equiv \forall s. \langle P \rangle \, s \leq \langle Q \rangle \, s \)

Note: \( \wp (a \sqcap b) \langle Q \rangle = \min (\wp a \langle Q \rangle) (\wp b \langle Q \rangle) \).
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It is the pessimistic expected value of the postcondition.
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These quantitative predicates are called expectations.
Probabilistic Choice

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\( a \frac{1}{2} \oplus b \) means ‘flip a coin — if heads \( a \) otherwise \( b \).’

What should \( \varphi (y := x^2 \frac{1}{2} \oplus y := 2x) (y = x) \) be?
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For an expectation, we’d take the weighted average:

\[
\wp(a p \oplus b) F = p \times \wp a F + (1 - p) \times \wp b F
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\( \wp (a \ p \oplus \ b) (y = x) \) is the probability that, if we start in state \( s \), \( y = x \) holds in the final state.

\( \wp (a \ p \oplus \ b) (y = x) 0 = 1 \) and \( \wp (a \ p \oplus \ b) (y = x) 1 = 1/2 \).

All other values are zero.
Combining Probability and Nondeterminism

How about this?

\[
E = \emptyset \left( (y := x^2 \ 1/2 \oplus y := 2x) \sqcap (y := x^2 \ 1/3 \oplus y := 2x) \right) (y = x)
\]

This time, \(E_0 = 1\) and \(E_1 = 1/3\).

\(E_x\) is the minimum probability that \(y = x\) will hold.
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$$E = \emptyset \left( (y := x^2 \ 1/2 \oplus y := 2x) \sqcap (y := x^2 \ 1/3 \oplus y := 2x) \right) (y = x)$$

Simply apply both rules:

$$E x = \min (1/2 \times \langle x = 0 \lor x = 1 \rangle + 1/2 \times \langle x = 0 \rangle)$$

$$\quad (1/3 \times \langle x = 0 \lor x = 1 \rangle + 2/3 \times \langle x = 0 \rangle)$$

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Combining Probability and Nondeterminism

How about this?

\[ E = \emptyset \left( (y := x^2 \cdot 1/2 \oplus y := 2x) \sqcap (y := x^2 \cdot 1/3 \oplus y := 2x) \right) (y = x) \]

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\( E x \) is the \textit{minimum} probability that \( y = x \) will hold.
These are basics of pGCL (Morgan & McIver, 2004).

It’s a formal model of computation incorporating probability and nondeterminism.

In the remainder of the talk I will introduce our mechanisation in Isabelle/HOL, and our work on the probabilistic verification of systems software.
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Expectations

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- \( \text{nneg } E \equiv \forall s. \ 0 \leq E \ s \)  
- \( \text{bounded } E \equiv \exists b. \ \forall s. \ E \ s \leq b \)
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The state space need not, in general, be finite.
Expectation Transformers

Programs are expectation transformers:

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\[ \forall P \, b. \text{bounded}_by \, b \, P \land \text{nneg} \, P \rightarrow \text{bounded}_by \, b \, (t \, P) \land \text{nneg} \, (t \, P) \]
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\[ \forall P \ Q. \ (\text{sound} \ P \land \text{sound} \ Q \land P \models Q) \rightarrow (t \ P) \vdash (t \ Q) \]

\[ \forall P \ c \ s. \ (\text{sound} \ P \land 0 < c) \rightarrow c \times t \ P \ s = t \ (\lambda s. \ c \times P \ s) \ s \]
A few primitives

\[ \text{Abort} \equiv \lambda ab \, P \cdot \text{if } ab \text{ then } \lambda s. 0 \text{ else } \lambda s. \text{bound_of } P \]

We model both strict (WP) and liberal (WLP) semantics. All these primitives are healthy.
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a \sqcap b \equiv \lambda ab P s. \min(a\ ab\ P\ s)\ (b\ ab\ P\ s)
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a \sqcap b \equiv \lambda a b P s. \text{min } (a \ ab \ P \ s) (b \ ab \ P \ s)
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\[
a \oplus b \equiv \lambda a b P s. p \times (a \ ab \ P \ s) + (1 - p) \times (b \ ab \ P \ s)
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\begin{align*}
\text{Abort} & \equiv \lambda ab P. \text{if } ab \text{ then } \lambda s. 0 \text{ else } \lambda s. \text{bound}_\text{of} P \\
\text{a} \sqcap \text{b} & \equiv \lambda ab P s. \text{min} (a \text{ ab } P s) (b \text{ ab } P s) \\
\text{a} \oplus \text{b} & \equiv \lambda ab P s. \text{p} \times (a \text{ ab } P s) + (1 - \text{p}) \times (b \text{ ab } P s) \\
\lozenge \text{a} & \equiv \text{a} \text{ True} \\
\end{align*}

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All these primitives are healthy.
The shallow embedding makes it easy to embed the L4.verified nondeterministic monad:

\[
\text{Exec} :: (\sigma \Rightarrow (\alpha \times \sigma) \text{ set}) \Rightarrow \text{bool} \Rightarrow (\sigma \Rightarrow \mathbb{R}) \Rightarrow \sigma \Rightarrow \mathbb{R}
\]

\[
\text{Exec } M \equiv \lambda ab R s. \text{glb } \{ R \text{ (snd } sa) . \text{ sa } \in \text{ M } s \}
\]
Embedding a Monad

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\[ \text{Exec} M \equiv \lambda ab R s. \text{glb} \{ R (\text{snd} sa). \; sa \in M s \} \]

We lift Hoare triples to probabilistic entailments:

\[
\begin{align*}
\text{WP}_\text{EXEC} & \quad \{ P \} \; \text{prog} \{ \lambda r \; s. \; Q \; s \} \quad \forall s. \; \text{prog} \; s \neq \{ \} \quad \exists s. \; P \; s \\
\langle P \rangle & \vdash \emptyset \; \text{prog} \; \langle Q \rangle
\end{align*}
\]
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\[ a \sqsubseteq b \quad E \vdash \wp.a.F \quad \therefore \quad E \vdash \wp.b.F \]
One of the principle tools in verification is *refinement*. A refinement relation allows us to transfer properties from *specification* to *implementation*:

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\end{align*}
\]

Given \( E \), if \( a \) establishes \( F \), then so does \( b \) or:

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\lozenge a.F \leq \lozenge b.F
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A refinement relation allows us to transfer properties from specification to implementation:

\[
\begin{align*}
\exists \ a \sqsubseteq b \quad & E \vdash \mathfrak{g}.a.F \\
& \quad \frac{E \vdash \mathfrak{g}.b.F}{\exists \ a \sqsubseteq b \quad & E \vdash \mathfrak{g}.a.F}
\end{align*}
\]

Given \( E \), if \( a \) establishes \( F \), then so does \( b \) or:

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\mathfrak{g}.a.F \leq \mathfrak{g}.b.F
\]

In pGCL, an implementation establishes any property with at least as great a probability as its specification.
Lattice Scheduling

An approach to efficiently eliminating leaks through shared state e.g. caches.

Only switch to a domain with higher clearance, or to the downgrader, which clears the cache:
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\text{scheduleL} \equiv \text{cd} :\in \\lambda s. \{n| (\text{cd}, n) \in S\}
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Lattice Scheduling

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The security property:

\[ \forall c, n. (c, n) \in S \rightarrow \text{sec_class.c} \leq \text{sec_class.n} \lor n = \text{downgrader} \]
Unfairness

A single-period schedule cannot include both $L_a$ and $L_b$. A nondeterministic scheduler might simply always pick $L_b$. 

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Randomised Lattice Scheduling

We’d still like to have asymptotic fairness between domains.
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If the matrix \( T \) satisfies:

\[
\forall c n. 0 < T (c, n) \rightarrow (c, n) \in S
\]

we have refinement, \( \text{scheduleL} \sqsubseteq \text{scheduleR} \).
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This scheduler is a Markov process, and if \( T \) is irreducible and positive recurrent, there exists a stationary distribution.
An efficient implementation might use a lottery:

```haskell
scheduleM t ≡ do
  c ← gets cd; l ← gets lottery;
  let n = l c t in modify(λs. s(cd := n))
od
```

The lottery has type: domain ⇒ word32 ⇒ domain.
Lottery Scheduling

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  od`

The lottery has type: `domain ⇒ word32 ⇒ domain`.

We chain in probability from above:

`scheduleC ≡ t from UNIV at 2^{-32} in Exec (scheduleM t)`
Data Refinement

We cannot show that scheduleR ⊑ scheduleC, as they operate on different state spaces:

```plaintext
record stateA = cd :: domain
record stateC = cd :: domain,
    lottery :: domain ⇒ word32 ⇒ domain
```

The lottery is an implementation detail, only cd matters.

Take the natural projection:

φ :: stateC ⇒ stateA.
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```

The lottery is an implementation detail, only \( cd \) matters. Take the natural projection: \( \phi :: \text{stateC} \Rightarrow \text{stateA} \).
We define \textit{data refinement}, \( \sqsubseteq_{\phi, \text{Pre}} \):\[
\begin{align*}
\begin{array}{c}
a \sqsubseteq_{\phi, \text{Pre}} b \\
E \vdash \Diamond a F \\
\text{Pre } s
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\begin{array}{c}
(E \circ \phi) s \vdash \Diamond b (F \circ \phi) s
\end{array}
\end{align*}
\]
If the ticket distribution represents the transition matrix:
\[
LR s \equiv \forall c, n. T(c, n) = \sum_{t} \text{lottery}_{s c t} = n^2 - 32
\]
we have another refinement step:
\[
scheduleL \sqsubseteq scheduleR \sqsubseteq \phi, LR scheduleC
\]
Data Refinement

We define data refinement, $\sqsubseteq_{\phi, \text{Pre}}$:

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\frac{a \sqsubseteq_{\phi, \text{Pre}} b}{E \vdash \wp a F \quad \text{Pre } s} \quad \frac{E \circ \phi}{(E \circ \phi) s \vdash \wp b (F \circ \phi) s}
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If the ticket distribution represents the transition matrix:

$$LR s \equiv \forall c, n. \ T(c, n) = \sum_{t. \ \text{lottery } s \ c \ t=n} 2^{-32}$$

we have another refinement step:
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&\quad (E \circ \phi) s \vdash \phi &\quad b \quad (F \circ \phi) s
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Finally, we attach a kernel model:

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We need only a few high-level properties, including:

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\{ cd = d \} \text{callKernel} \{ cd = d \}
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which is a specification in the L4.verified Hoare logic, from which we establish:
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The kernel may modify the lottery!
If the kernel additionally preserves the lottery relation:

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The kernel implements a fair, secure scheduler.
Summary

We have:

• Motivated probabilistic verification for systems.
• Mechanised pGCL in Isabelle/HOL.
• Verified a randomised scheduler.
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Questions?